§1. 周炜良引理

Thm (Chow's lemma). $S = \mathrm{Spec}(A = \text{Noeth. ring})$, $X \xrightarrow{f} S$ sep. + f.type. + X int'l

$$\exists \quad \begin{tikzcd} & X' \arrow[dl, "\pi"', "\text{proper}"] \arrow[dr, "i", "\text{imm.}"'] & \\ X \arrow[dr, "f"'] & & \mathbb{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

int'l

& $\exists\ U \subseteq X$ dense open s.t.

$$\pi^{-1}(U) \xrightarrow{\cong} U.$$

(f proper $\Rightarrow$ i is a closed imm.)

pf: $X = \bigcup_{i=1}^{m} U_i$, $U_i = \mathrm{Spec}(B_i)$, $U := \bigcap_{i=1}^{m} U_i = \mathrm{Spec}(B)$.

For each i,

$$\begin{tikzcd} & \overline{U_i} \arrow[dr, hook] & \\ \mathrm{Spec}(B_i) \arrow[ur, "j_i", "\subseteq"'] \arrow[rr, "h_i"] \arrow[dr, hook] & & \mathbb{P}^{n_i}_S \\ & \mathbb{A}^n_S \arrow[ur, "\subseteq"'] & \end{tikzcd}$$

int'l

by lemma before (h_i is q-c. & U_i is reduced).

$$\begin{tikzcd} & U^* \arrow[dr, hook] & & \\ U \arrow[ur, "\text{open}", "\subseteq"'] \arrow[r, "(j_i|_U)", "\text{imm.}"'] \arrow[d, "\Delta"] & \mathbb{P}^{n_1}_S \times_S \mathbb{P}^{n_2}_S \times_S \cdots \times_S \mathbb{P}^{n_m}_S \arrow[r, hook, "\text{Segre}"] & \mathbb{P}^{\prod(n_i+1)-1}_S \\ U \times_S U \times_S \cdots \times_S U \subseteq \prod U_i \arrow[ur] & & \end{tikzcd}$$

int'l

$$\begin{tikzcd} U \subseteq V_i \subseteq \overline{U} \arrow[r, hook] \arrow[d, "pr_i"] & \prod_{i=1}^{m} \mathbb{P}^{n_i}_S \arrow[d, "pr_i"] \\ U_i \subseteq \overline{U_i} \arrow[r, hook] & \mathbb{P}^{n_i}_S \end{tikzcd}$$

$U \subseteq U_i$, $V_i \xrightarrow{pr_i} U_i$

$V_i \subseteq \overline{U}$

$V_i := pr_i^{-1}(U_i)$.

$U \subseteq X' := \bigcup_{i=1}^{m} V_i \subseteq U^*$

$V_i \longrightarrow U_i \subseteq X$ glue to get $X' \xrightarrow{\pi} X$: $pr_i|_{V_i \cap V_j} = pr_j|_{V_i \cap V_j}$

as they agree on $U \subseteq V_i \cap V_j$.

($\mathrm{Eq}(V_i \cap V_j \rightrightarrows X) \cong (V_i \cap V_j) \times_{X \times_S X} X$

is a closed subsch. in $V_i \cap V_j$ which contains U)

(=) clopen & dense between int'l schemes

$$\begin{tikzcd} V_i \overset{(=)}{\subseteq} \pi^{-1}(U_i) \arrow[d, "\pi"] \\ U_i \end{tikzcd} \qquad V_i \xrightarrow[\text{proper}]{pr_i} U_i$$

$\Longrightarrow$ π is proper + surj.

Similarly

$$\begin{tikzcd} U \overset{(=)}{\subseteq} \pi^{-1}(U) \arrow[d, "\pi"] \\ U \end{tikzcd}$$

Cor. $f: X \xrightarrow{\text{proper}} Y$, $\mathcal{F} \in Coh(X)$, Y (hence X also) loc. Noeth.

Then $f_*\mathcal{F} \in Coh(Y)$.

pf: WMA: $Y = Spec(A)$.

$\exists$ fil. $0 \subseteq \mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_m = \mathcal{F}$, $Z_i \overset{\text{closed}}{\hookrightarrow} X$ int'l, $\mathcal{I}_i \subseteq \mathcal{O}_{Z_i}$.

s.t. $\mathcal{F}_i / \mathcal{F}_{i-1} = (Z_i \hookrightarrow X)_* \mathcal{I}_i$.

$$0 \to \Gamma(\mathcal{F}_{i-1}) \to \Gamma(\mathcal{F}_i) \to \Gamma((Z_i \hookrightarrow X)_* \mathcal{I}_i)$$
$$\| $$
$$\Gamma(Z_i, \mathcal{I}_i) \hookrightarrow \Gamma(Z_i, \mathcal{O}_{Z_i}) \quad (\hookrightarrow \Gamma(Z_i', \mathcal{O}))$$

So suffices, by induction, to show $\underbrace{\Gamma(Z_i, \mathcal{O}_{Z_i})}$ fin. A-mod.

Now apply 周引理
$$\begin{tikzcd} & Z_i' \arrow[dl, "\pi"'] \arrow[d, "proj"] \\ Z_i \arrow[r] & Y \end{tikzcd}$$
, $\mathcal{O}_{Z_i} \hookrightarrow \pi_* \mathcal{O}_{Z_i'}$.

Appeal to $proj_* \mathcal{O} \in Coh(Y)$.

§2. Formal function Thm.

$$X \xrightarrow{f} Y \ni y \rightsquigarrow \begin{tikzcd} X \\ X \times_Y Spec(\mathcal{O}_{Y,y}) \arrow[u, "pr_1"] \arrow[r, "pr_2"] & Spec(\mathcal{O}_{Y,y} =: A) \end{tikzcd} \qquad \mathfrak{m} \subset A$$

Lemma: f qc qs, $\mathcal{F} \in QCoh(X)$, $(f_*\mathcal{F})_y \cong \Gamma(X \times_Y Spec(A), pr_1^*\mathcal{F})$.

2 completions: $((f_*\mathcal{F})_y)^\wedge_{\mathfrak{m}} := \lim_n \Gamma(X \times_Y Spec(A), pr_1^*\mathcal{F}) / \mathfrak{m}^n$.

always arrow $\downarrow$ Comp$(f, \mathcal{F}, y)$

$$\lim_n \Gamma(X \times_Y Spec(A/\mathfrak{m}^n), pr_1^*\mathcal{F})$$

Thm* (FFT): If f proper, Y (loc.) Noeth., $\mathcal{F}$ coh., then Comp$(f, \mathcal{F}, y)$ is an isom.

§3. Stein factorization.

Thm: S = loc. Noeth., $X \xrightarrow{f} S$ proper, set

$$\begin{tikzcd} X \arrow[rr, "f'"] \arrow[dr, "f"'] & & \mathrm{Spec}\, f_* \mathcal{O}_X =: S' \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

Then

(1) π finite

(2) $f'_* \mathcal{O}_X = \mathcal{O}_{S'}$

(3) f' is proper

(4) f' has conn'd fibers. (note that $\emptyset$ is NOT conn'd.)

pf: (1)–(3) exercise.

For (4): $f_* \mathcal{O}_X = \mathcal{O}_S \Rightarrow$ conn'd fiber.

Suppose $X_s = Y \amalg Z$, then so is $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}^n)$

$X_s = X \times_S \mathrm{Spec}(\kappa(s))$ $\qquad \simeq Y_n \amalg Z_n$.

(b/c top. space agrees).

$$\Rightarrow \quad (f_* \mathcal{O}_X)^\wedge_s = \mathcal{O}_{S,s}^\wedge$$

$$\underset{\text{FFT}}{=} \quad \lim_n \Gamma(Y_n \amalg Z_n, \mathcal{O}) = \text{nonzero ring} \times \text{nonzero ring}.$$

$\times$.

Cor. (Baby version of ZMT).

$X \xrightarrow{f} Y$, proper between loc. Noeth. integral schemes. Assume:

(birat'l) (1) $f|_{\eta_X} : \eta_X \xrightarrow{\cong} \eta_Y$ $\quad (\Leftrightarrow (1'): \exists\, U \subseteq Y$ s.t. $f^{-1}(U) \xrightarrow{\cong} U)$.

(2) Y is normal. ($\mathcal{O}_{Y,y}$ are int'lly closed domains $\forall y \in Y$).

Then f has conn'd fibers.

pf: $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ finite + inj. + same function field,
hence $=$ by normality of Y.